Five-dimensional SYM from undeformed ABJM

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Abstract

We expand undeformed ABJM theory around the vacuum solution that was found in arxiv:0909.3101. This solution can be interpreted as a circle-bundle over a two-dimensional plane with a singularity at the origin. By imposing periodic boundary conditions locally far away from the singularity, we obtain a local fuzzy two-torus over which we have a circle fibration. By performing fluctuation analysis we obtain five-dimensional SYM with the precise value on the coupling constant that we would obtain by compactifying multiple M5 branes on the vacuum three-manifold. In the resulting SYM theory we also find a coupling to a background two-form.

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1 Introduction

In [1] a vacuum solution was found in undeformed ABJM theory, simply by solving the equation of motion in a static field configuration. We will denote the four complex scalar fields in ABJM theory as Z^A for $A = 1, 2, 3, 4$. We split $A = (a, \dot{a})$ where $a = 1, 2$ and $\dot{a} = \dot{1}, \dot{2}$. The gauge group is $U(N) \times U(N)$ and Z^A are $N \times N$ bifundamental matrices. However, by utilizing the star-product, we can map these matrices into functions living on a fuzzy two-torus. In the large N limit, we have to leading order that the star-product is just the usual product of functions, and the star-commutator has the leading term which is the Poisson bracket. In this limit the solution that was obtained in [1] can be presented as $Z^A = T^A$ where

$$\begin{aligned} T^a &= x^a e^{i\psi} \\ T_{\dot{a}} &= 0 \end{aligned}$$

Here x^a span a two-dimensional plane, and ψ is a coordinate on a circle fiber over this plane. This solution is a three-manifold M_3 with metric

$$ds^2 = \delta_{ab} dx^a dx^b + r^2 d\psi^2$$

where $r = \sqrt{x^a x^a}$. The scalar curvature is $\mathcal{R} = 1/r$ which is singular at $x^a = 0$.

It is true that M_3 does not seem to be translationally invariant, but M_3 is translationally invariant in the spacetime of ABJM theory. Since a constant shift of the fermions in ABJM theory is an additional supersymmetry of the Lagrangian, usually refered to as a kinematic supersymmetry [3], we deduce that the solution is maximally supersymmetric from the point of view of ABJM theory.

However from the M5 brane point of view, where the M5 brane worldvolume is $\mathbb{R}^{1,2} \times M_3$, it is unclear to us whether we can have maximal supersymmetry. On M_3 we can only find two independent Killing vectors (instead of six as would have been the case had M_3 been maximally symmetric)

$$\begin{aligned} V_1 &= x^2 \partial_1 - x^1 \partial_2 + \arctan \frac{x^2}{x^1} \partial_\psi \\ V_2 &= \partial_\psi \end{aligned}$$

This means that we can not hope to close untwisted $(2, 0)$ supersymmetry variations on the M5 brane into Lie derivatives on M_3. Henceforth we will only study the bosonic part of the theory, and leave a possible supersymmetric twisted M5 theory for future studies.

A previous work which dealt with the emergence of the D4 brane from undeformed ABJM theory, is [2]. In this work a non-commutativity parameter is introduced by hand, and consequently the Yang-Mills coupling depends on a quantity which is not present in ABJM Lagrangian. In this paper we relate the non-commutativity parameter to parameters which are present in ABJM theory. That is, the rank of the gauge group N, and the Chern-Simons level K.

2 Triality of BLG theory

The R-symmetry of BLG theory is $SO(8)$ which has triality relating 8_v, 8_c and 8_c representations. The invariant quantities that carry all these representation

indices, are the $SO(8)$ gamma matrices,

$$\Gamma_I = \begin{pmatrix} 0 & \Gamma_{I\alpha\dot\beta} \\ \Gamma^{I\dot\alpha\beta} & 0 \end{pmatrix}$$

Hermiticity of the gamma matrices implies

$$\Gamma^*_{I\alpha\dot\beta} = \Gamma^{I\dot\beta\alpha} \tag{1}$$

Triality is a collection of six maps that permutes 8_v, 8_s and 8_c. We will study those trial maps which relate the ABJM and BLG theories. These act on the indices according to

$$\begin{aligned} I &\to \alpha \\ \alpha &\to \dot\beta \\ \dot\beta &\to I \end{aligned}$$

and its inverse obtained by reversing the directions of the arrows. Under the above triality map the half-gamma matrices transform according to

$$\begin{aligned} \Gamma^{I\dot\beta\alpha} &\to \Gamma^{\alpha I\dot\beta} \equiv \Gamma_{I\alpha\dot\beta} \\ \Gamma_{I\alpha\dot\beta} &\to \Gamma_{\alpha\dot\beta I} \equiv \Gamma^{I\dot\beta\alpha} \end{aligned}$$

To describe how the triality map acts on other quantities, we introduce three bosonic quantities U^α, $V^{\dot\alpha}$ and W_I that we subject to the constraints

$$\begin{aligned} U_\alpha &= \Gamma_{I\alpha\dot\beta} W^I V^{\dot\beta} \\ V^{\dot\alpha} &= \Gamma^{I\dot\alpha\beta} U_\beta W_I \\ W_I &= \Gamma_{I\alpha\dot\beta} V^{\dot\beta} U^\alpha \end{aligned}$$

and

$$\begin{aligned} U^\alpha U_\alpha &= 1 \\ V^{\dot\alpha} V_{\dot\alpha} &= 1 \\ W_I W^I &= 1 \end{aligned}$$

Then the triality maps read

$$\begin{aligned} X_\alpha &= \Gamma_{I\alpha\dot\beta} X^I V^{\dot\beta} \\ \psi^{\dot\alpha} &= \Gamma^{I\dot\alpha\beta} \psi_\beta W_I \\ \epsilon_I &= \Gamma_{I\alpha\dot\alpha} \epsilon^{\dot\alpha} U^\alpha \end{aligned}$$

and the inverse

$$\begin{aligned} X^I &= \Gamma^{I\dot\alpha\beta} X_\beta V_{\dot\alpha} \\ \psi_\alpha &= \Gamma_{I\alpha\dot\beta} \psi^{\dot\beta} W^I \\ \epsilon^{\dot\alpha} &= \Gamma^{I\dot\alpha\beta} \epsilon_I U_\beta \end{aligned}$$

Here X^I denote the eight scalar fields in 8_v, ψ_α denotes the 8_s-spinors and $\epsilon^{\dot\alpha}$ the supersymmetry parameter in 8_c. For single index quantities we use the convention that rising and lowering indices correspond to complex conjugation.

For multiple index quantities we have also to specify an ordering prescription when rising and lowering the indices under complex conjugation. An example of this is (1). It is true that we can stick to a basis where all entries are real in the gamma matrices and the Majorana spinors, but things get more transparent if we work in a general basis.

We have the following identities

$$\Gamma^{I\dot{\alpha}\beta}U_\beta U^\gamma \Gamma_{I\gamma\dot{\delta}} = \delta^{\dot{\alpha}}_{\dot{\delta}}$$

$$\Gamma_{J\alpha\dot{\gamma}}V^{\dot{\gamma}}V_{\dot{\beta}}\Gamma^{I\dot{\beta}\alpha} = \delta^I_J$$

$$W_I\Gamma^{I\dot{\beta}\alpha}\Gamma_{J\alpha\dot{\gamma}}W^J = \delta^{\dot{\beta}}_{\dot{\gamma}}$$

To prove the first of these identities we note the Fierz rearrangement

$$UU^\dagger = \frac{1}{16}\Big(U^\dagger U + \frac{1}{12}U^\dagger\Gamma^{IJKL}U\Gamma_{IJKL}\Big)(1+\Gamma)$$

together with the gamma matrix identity

$$\Gamma_I\Gamma_{JKLM}\Gamma^I = 0$$

To prove the second of these identities we use the Clifford algebra. The third identity follows by trace properties of the gamma matrices.

Armed with these identities we can map any contraction to its trial contraction, for example

$$\begin{aligned} X^\alpha Y_\alpha &= (\Gamma^{I\dot{\beta}\alpha}X_I V_{\dot{\beta}})(\Gamma_{J\alpha\dot{\gamma}}Y^J V^{\dot{\gamma}}) \\ &= \Gamma_{J\alpha\dot{\gamma}}V^{\dot{\gamma}}V_{\dot{\beta}}\Gamma^{I\dot{\beta}\alpha}X_I Y^J \\ &= X_I Y^I \end{aligned}$$

3 Relating BLG with ABJM

In string theory we are familiar with that two different theories can be unified if we move one dimension higher. In this section we will recall how BLG and ABJM theories, which are in general unrelated in two auxiliary dimensions (the only exception being when the gauge group is $SU(2)\times SU(2)$), get unified in three auxiliary dimensions where we can use a star-3-product [4].

In ABJM theory we have a three-bracket defined as [6]

$$[T^a, T^b; T^c] = T^a T_c T^b - T^b T_c T^a$$

The generators T^a are usually taken to be $N\times N$ matrices and the gauge group $U(N)\times U(N)$. We can also use star-products of functions. These functions live on a two-manifold. It is well-known how the star-product is mapped isomorphically to matrix multiplication when we have either a two-sphere or a two-torus, and in this paper we will only consider the two-torus. The idea is to describe a manifold by the algebra of functions. While it is true that the functions may live on a smooth and classical two-torus, if we only have a finite set of functions

we can not probe the smooth structure of the two-torus. Instead the torus will appear like a fuzzy or noncommutative manifold where the coordinates do not quite commute. The finite set of functions correspond to the finite rank of the corresponding matrices. By taking $N \to \infty$ we obtain the smooth manifold.

Let us now assume that we are given a three-manifold M_3 and some three-algebra generators T^a which are functions on M_3. We denote by $T_a = (T^a)^*$ the complex conjugated elements. In general the star-3-product defined as

$$T^a * T_c * T^b = \exp\left\{\frac{\hbar}{2}\sqrt{g}\epsilon^{\alpha\beta\gamma}\partial_\alpha\partial'_\beta\partial''_\gamma\right\} T^a(\sigma)T^b(\sigma')T_c(\sigma'')|_{\sigma=\sigma'=\sigma''}$$

is not associative. We will now assume that M_3 is a circle bundle and let

$$\sigma^\alpha = (\sigma^a, \psi = \sigma^3)$$

be coordinates on M_3, σ^a (for $a = 1, 2$) be coordinates on a two-dimensional base-manifold, and ψ be a coordinate on the fiber. It is necessary that we restrict ourselves to functions on M_3 which are on the form

$$T^a = e^{i\psi}\widetilde{T}^a(\sigma^a) \tag{2}$$

in order for the star-3-product to become associative. We denote by $g_{\alpha\beta}$ the metric on the three-manifold, and by G_{ab} the metric on the base manifold. We define the totally antisymmetric tensors like $\epsilon_{123} = 1$ and rise all indices by the inverse metrics, so for example $g\epsilon^{123} = 1 = G\epsilon^{12}$. We define a star-2-product as

$$T^a * T^b = \exp\left\{\frac{i\mathcal{E}}{2}\sqrt{G}\epsilon^{ab}\partial_a\partial'_b\right\} T^a(\sigma)T^b(\sigma')$$

The relation between $\mathcal{E}$ and $\hbar$ reads

$$\mathcal{E} = \hbar\sqrt{\frac{G}{g}}$$

The associated star-commutator is given by

$$\begin{aligned} [f, g] &= i\epsilon\{f, g\} + \mathcal{O}(\epsilon^2) \\ \{f, g\} &= \sqrt{G}\epsilon^{ab}\partial_a f\partial_b g \end{aligned}$$

The star-3-product we use is not a genuine star-3-product since we restrict ourselves to functions that are essentially living on the base manifold, all having the same rather trivial dependence on the fiber according to Eq (2). Moreover, just as one should expect of such a star-3-product, it can be expressed as a composition of two consecutive star-2-products,

$$T^a * T_c * T^b = (T^a * T_c) * T^b$$

We define a totally antisymmetric three-bracket as

$$[T^a, T^b, T_c] = T^a * T_c * T^b - T^b * T_c * T^a$$

We will refer to this bracket as the star-3-commutator. To first order in $\hbar$ it is given by

$$[T^a, T^b, T_c] = \hbar\{T^a, T^b, T_c\}$$

In an appendix in [5] it is shown that the star-3-commutator is totally antisymmetric to all orders.

In BLG theory we need a totally antisymmetric three-bracket. As we have shown, we may use the star-3-commutator on a certain two-dimensional subset of functions on a circle-bundle over a two-manifold. By triality of $SO(8)$ we may always assume that the field content of BLG theory consists of eight scalars X_α in 8_s and eight fermions $\psi^{\dot{\alpha}}$ in 8_c. The sextic potential is given by

$$V = \frac{1}{12}|[X^\alpha, X^\beta, X^\gamma]|^2$$

To connect with ABJM theory we decompose the scalar fields as

$$X^\alpha = \begin{pmatrix} Z^A(x^a)e^{i\psi} \\ Z_A(x^a)e^{-i\psi} \end{pmatrix}$$

This decomposition breaks $SO(8)$ down to $SO(6)$ whereof Z^A is a Weyl spinor. Though the more common way of expressing the same thing is as the defining representation of $SU(4) \simeq SO(6)$. By utilizing the total antisymmetry of the three-bracket, we expand out the sextic potential as

$$V = \frac{1}{12}\left(2|[Z^A, Z^B, Z^C]|^2 + 6|[Z^A, Z^B, Z_C]|^2\right)$$

By using the fundamental identity we derive the identity

$$|[Z^A, Z^B, Z^C]|^2 = |[Z^A, Z^B, Z_C]|^2 - 2|[Z^A, Z^B, Z_B]|^2$$

and we find the ABJM sextic potential

$$V = \frac{2}{3}\left(|[Z^A, Z^B, Z_C]|^2 - \frac{1}{2}|[Z^A, Z^B, Z_B]|^2\right)$$

We may restore the ABJM three-bracket and we get

$$V = \frac{2}{3}\left(|[Z^A, Z^B; Z^C]|^2 - \frac{1}{2}|[Z^A, Z^B; Z^B]|^2\right)$$

Our first observation now is that only Z^A occurs in this final expression, and no Z_A. This means that only T^a three-algebra generators arise in this expression, and no T_a. By taking $T^a = e^{i\psi}\widetilde{T}^a$, the ABJM three-bracket reduces as

$$[T^a, T^b; T^c] = T_c[T^a, T^b] + [T^a, T_c]T^b - [T^b, T_c]T^a$$

In the RHS we have usual star-product multiplications (star-2-products). Mapping these to matrix multiplications, we make contact with ABJM theory as it was originally formulated.

We conclude that the bosonic part of the ABJM Lagrangian can be expressed as

$$\mathcal{L} = \frac{KN}{2\pi}\left(\mathcal{L}_{kin} + \mathcal{L}_{CS} + \mathcal{L}_{pot}\right)$$

where

$$\mathcal{L}_{kin} = -\frac{1}{2}|D_\mu X^\alpha|^2$$

$$
\begin{aligned}
\mathcal{L}_{CS} &= \frac{1}{2}\epsilon^{\mu\nu\lambda}\Big(\left\langle T^b, [T^c, T^d; T^a]\right\rangle A_\mu{}^c{}_b \partial_\nu A_\lambda{}^d{}_a \\
&\quad -\frac{2}{3}\left\langle [T^a, T^c; T^d], [T^f, T^b; T^e]\right\rangle A_\mu{}^b{}_a A_\nu{}^d{}_c A_\lambda{}^f{}_e\Big) \\
\mathcal{L}_{pot} &= -\frac{1}{12}|[X^\alpha, X^\beta; X^\gamma]|^2
\end{aligned}
$$

4 The M5 brane solution

We will now review the solution that we presented in the introduction, closely following the original work [1]. We decompose the ABJM scalar fields as

$$
Z^A = \begin{pmatrix} Z^a \\ Z_{\dot{a}} \end{pmatrix}
$$

corresponding to $SU(4) \to SU(2) \times SU(2)$, and make the following ansatz for these components,

$$
\begin{aligned}
Z^a &= x^a e^{i\psi} \\
Z_{\dot{a}} &= 0
\end{aligned} \tag{3}
$$

Here

$$
0 \leq \psi \leq \frac{2\pi}{K}
$$

and x^a are real. We assume that the metric on the base manifold is given by

$$
ds^2 = \delta_{ab} dx^a dx^b
$$

and we define the metric tensor $G_{ab} = \delta_{ab}$ and its determinant is $G = 1$.

By solving the equation of motion we will now determine the metric on the three-manifold M_3, whose base manifold is the two-dimensional plane described above. We make the ansatz

$$
ds^2 = \delta_{ab} dx^a dx^b + f(x^a) d\psi^2
$$

If we let $x^\alpha = (x^a, \psi)$ denote the coordinates on M_3 the ansatz for the metric tensor reads

$$
g_{\alpha\beta} = \begin{pmatrix} \delta_{ab} & 0 \\ 0 & f \end{pmatrix}
$$

and its determinant is $g = f$. Here f is a function that is to be determined by solving the static equation of motion of ABJM theory.

Inserting our ansatz into the sextic potential (ignoring any overall factor)

$$
|\left[Z^A, Z^B; Z^C\right]|^2 - \frac{1}{2}|\left[Z^A, Z^B; Z^B\right]|^2
$$

it reduces to

$$
|\left[x^1, (x^2)^2\right]|^2 + |\left[x^2, (x^1)^2\right]|^2
$$

We now vary x^1. This gives us the equation of motion

$$
\left[(x^2)^2, \left[x^1, (x^2)^2\right]\right] + \left(x^1, \left[x^2, \left[(x^1)^2, x^2\right]\right]\right) = 0
$$

and a corresponding equation obtained by exchanging indices 1 and 2.

The noncommutativity parameter that sits in the star-2-product becomes

$$\mathcal{E} = \frac{\hbar}{\sqrt{f}}$$

To lowest order in this parameter, the star-2-(anti-)commutator is given by

$$\begin{aligned} [x^1, x^2] &= \frac{i\hbar}{\sqrt{f}}\{x^1, x^2\} \\ (x^1, x^2) &= 2x^1 x^2 \end{aligned}$$

where the Poisson bracket is given by

$$\{x^1, x^2\} = 1$$

We now get the equation of motion as

$$\left((x^1)^2 + (x^2)^2\right)[x^2, [x^1, x^2]] - x^1[x^1, x^2]^2 = 0$$

By further noting that

$$[x^2, \bullet] = -\frac{i\hbar}{\sqrt{f}}\frac{\partial}{\partial x^1}$$

we can express the equation of motion as

$$\left((x^1)^2 + (x^2)^2\right)\frac{\partial}{\partial x^1}\left(\frac{1}{\sqrt{f}}\right) + x^1\frac{1}{\sqrt{f}} = 0$$

We have a similar equation from varying Z^2 which is obtained by exchanging indices 1 and 2. The solution to these two equations is given by

$$f = C\left((x^1)^2 + (x^2)^2\right)$$

and thus we deduce that any three-manifold on with the metric

$$ds^2 = (dx^1)^2 + (dx^2)^2 + C\left((x^1)^2 + (x^2)^2\right)d\psi^2$$

for any constant C, solves the ABJM equation of motion. But we can absorb this constant into ψ by rescaling ψ. Hence we can always assume that the metric is given by

$$ds^2 = (dx^1)^2 + (dx^2)^2 + r^2 d\psi^2 \tag{4}$$

where we define

$$r^2 = (x^1)^2 + (x^2)^2$$

This now, is the metric that is induced from the flat metric on $\mathbb{C}^4/\mathbb{Z}_K$,

$$ds^2 = dZ^a dZ_a + dZ_{\dot{a}} dZ^{\dot{a}}$$

We now also see that we shall let $0 \leq \psi \leq \frac{2\pi}{K}$ if we consider the orbifold $\mathbb{C}^4/\mathbb{Z}_K$. This M5 brane solution is valid for any integer value on K.

If we parameterize the base two-manifold by the coordinates (x^1, x^2) in which the metric is a given above, then the non-commutativity parameter in the star-2-product is given by

$$\mathcal{E} = \frac{\hbar}{r}$$

and is clearly non-constant.

The star-2-product is still associative and the Jacobi identity is still satisfied, even for a non-constant non-commutativity parameter. To check this, we just have to notice that $\epsilon^{a[b}\epsilon^{cd]} = 0$, which is true in two dimensions since we antisymmetrize over three indices.

The three-manifold M_3 can alternatively be expressed as an embedded surface in $\mathbb{C}^2$ as

$$T^1 T_2 - T^2 T_1 = 0$$

where $T_a = (T^a)^*$. The solution obeys the three-algebra

$$\begin{aligned} \{T^1, T^2, T_2\} &= -2iT^2 \\ \{T^2, T^1, T_1\} &= 2iT^1 \end{aligned}$$

We can bring this into the standard form of $SO(4)$ three-algebra by defining

$$\begin{aligned} S^1 &= T^1 + iT^2 \\ S^2 &= T^1 - iT^2 \end{aligned}$$

Then we find

$$\{S^a, S^b, S_c\} = -8\delta^{ab}_{cd} S^d$$

which is the standard $SO(4)$ three-algebra expressed in a complex basis. We have difficulties finding finite-dimensional matrix representations of this algebra which also satisfy the condition

$$S^1 S_1 - S^2 S_2 = 0$$

which describes the embedded three-manifold M_3 in these new coordinates.

Finally we can express M_3 as the embedding

$$y = 0$$

by choosing the coordinates as

$$z^a = \left(x^a + iy\frac{\epsilon^{ab} x_b}{r}\right) e^{i\psi}$$

In polar coordinates

$$re^{i\varphi} = x^1 + ix^2$$

we have the coordinate transformation

$$s^1 = (r + y)e^{i(\psi + \varphi)}$$

$$s^2 = (r-y)e^{i(\psi-\varphi)}$$

where thus $s^a|_{y=0} = S^1$, and the metric becomes

$$\begin{aligned} ds^2 &= \frac{1}{2}\left(|ds^1|^2 + |ds^2|^2\right) \\ &= dr^2 + dy^2 + (r^2+y^2)(d\psi^2 + d\varphi^2) + 2ryd\psi d\varphi \end{aligned}$$

Transformed back to cartesian coordinates on the 2-plane, this reads

$$ds^2 = \delta_{ab}dx^a dx^b + r^2 d\psi^2 + dy^2 + \mathcal{O}(y)$$

We see that y is a normal direction to M_3

$$T^a = z^a|_{y=0}$$

and there is no off-diagonal metric components along the y-direction,

$$\begin{aligned} g_{ya} &= 0 \\ g_{y\psi} &= 0 \end{aligned}$$

when we confine ourselves to M_3.

5 Local quantization of the solution

We have not managed to quantize M_3. But also, we can not quantize $\mathbb{R}^2$ since this a non-compact space. What we can do is to consider a local two-torus somewhere on $\mathbb{R}^2$, far away from the curvature singularity at the origin. To this end we will express the vacuum solution as

$$T^a = (v^a + R\sigma^a)\, e^{i\psi} \tag{5}$$

and we will assume that $v^a >> R$ and let $0 \leq \sigma^a \leq 2\pi$ parametrize the local two-torus. Here R is a length scale of this two-torus. The metric is

$$ds^2 = R^2 \delta_{ab} d\sigma^a d\sigma^b + v^2 \left(1 + \mathcal{O}\left(\frac{R}{v}\right)\right) d\psi^2$$

where $v = \sqrt{\delta_{ab}v^a v^b}$. We will treat $\frac{R}{v}$ as an expansion parameter, which will enable us to perform a systematic fluctuation analysis to obtain the D4 brane Lagrangian. We have the square root determinants of the metrics

$$\begin{aligned} \sqrt{G} &= R^2 \\ \sqrt{g} &= R^2 v \left(1 + \mathcal{O}\left(\frac{R}{v}\right)\right) \end{aligned}$$

The relation between the two-dimensional and three-dimensional non-commutativity parameters then reads

$$\mathcal{E} = \frac{\hbar}{v}\left(1 + \mathcal{O}\left(\frac{R}{v}\right)\right)$$

and the quantization condition for the two-dimensional non-commutativity parameter can be inferred from the fuzzy two-torus structure[2]

$$\mathcal{E} = \frac{2\pi R^2}{N}$$

where N is the size of the corresponding matrix realization of the fuzzy two-torus.

It is important to note that the background is the two-torus T^2 and not the point $Z^a = v^a e^{i\psi}$. But this may at first sight seem confusing since then $Z^a = Z^a(\sigma)$ are functions rather than taking specific values as is the usual situation when one gives a vacuum expectation value to a scalar field. But here the scalar fields defined by Eq (5) on the whole T^2 is really our vacuum expectation value. The intuitive picture is that the vacuum expectation value is an infinite-dimensional diagonal matrix whose eigenvalues are the different points on T^2. At each point we have an M2 brane with worldvolume $\mathbb{R}^{1,2}$, and hence the collection of all these M2 branes give us a D4 brane whose worldvolume is $\mathbb{R}^{1,2} \times T^2$. This picture is somewhat intuitive though. In reality we have a finite set of M2 branes and a fuzzy T^2. Since it is fuzzy, we have a Heisenberg type uncertainty forbidding us to pack the M2 branes too dense, thus there is room only for a finite number of M2 branes on the fuzzy T^2.

Since $\sigma^a \sim \sigma^a + 2\pi$, we also find that the T^a are compact. Accordingly we shall also take the ABJM scalar fields to be compact,

$$Z^a \sim Z^a + 2\pi R$$

We will for the most part of this paper consider only small fluctuations around (5), much smaller than the size of T^2, and so the fact that these scalar fields are compact can be largely ignored.

6 The Higgs mechanism

To study the Higgs mechanism, the first thing we need to do is to consider the covariant derivative acting on a scalar field that we will eventually give a vacuum expectation value. The covariant derivative is given by

$$D_\mu Z^A = \partial_\mu Z^A + [Z^A, T^a; T^b] A_\mu{}^b{}_a$$

We define

$$\begin{aligned} A^- &= \frac{i}{2}[T^a, T_b] A_\mu{}^b{}_a \\ A^+ &= \frac{i}{2}(T^a, T_b) A_\mu{}^b{}_a \end{aligned}$$

where we use round brackets for the anticommutator. We now get

$$D_\mu Z^A = \partial_\mu Z^A + i[A_\mu^+, Z^A] + i(Z^A, A_\mu^-)$$

[2]A definition of the fuzzy two-torus in the conventions of this paper is found in [5]. A review paper of fuzzy manifolds is [11] where also many references can be found to fuzzy Riemann manifolds.

We give a Higgs vacuum expectation value to the scalar fields,

$$Z^A = T^A + Y^A$$

Here T^A is the vacuum expectation value, and Y^A are the fluctuations. We then expand the covariant derivative, and find

$$D_\mu Z^A = D_\mu Y^A + i(T^A, A_\mu^-) + i[A_\mu^+, T^A]$$

where we define

$$D_\mu Y^A = \partial_\mu Y^A + i[A_\mu^+, Y^A]$$

where, if we can neglect $i(Y^A, A_\mu^-)$, we have isolated the term that involves A_μ^- from all the rest. Eventually we will discover that A_μ^- enters the Lagrangian only algebraically and can be integrated out.

We define the induced metric tensor as

$$\begin{aligned} G_{ab} &= \partial_{(a} T^A \partial_{b)} T_A \\ G_{IJ} &= \partial_{(I} T^A \partial_{J)} T_A \end{aligned}$$

7 Fluctuation analysis

We will now expand the ABJM Lagrangian about the vacuum solution where we keep only the leading order terms in the expansion parameter $\frac{R}{v}$.

7.1 The kinetic term

We expand out the kinetic term

$$\mathcal{L}_{kin} = -\left\langle D_\mu Z^A, D_\mu Z^A \right\rangle$$

around the Higgs vacuum expection value. We define

$$\begin{aligned} Y^A &= Y^a \partial_a T^A + Y^I \partial_I T^A \\ Y^a &= \lambda \sqrt{G} \epsilon^{ab} A_b \\ Y^I &= \lambda \phi^I \end{aligned}$$

and define

$$A_\mu^+ = -\frac{\lambda}{\mathcal{E}} A_\mu$$

We then get

$$D_\mu Z^A = \lambda \left(\sqrt{G} \epsilon^{ab} F_{\mu b} \partial_a T^A + D_\mu \phi^I \partial_I T^A \right) + 2i T^A A_\mu^-$$

where

$$\begin{aligned} F_{\mu a} &= \partial_\mu A_a - \partial_a A_\mu - \frac{i\lambda}{\mathcal{E}} [A_\mu, A_a] \\ D_\mu \phi^I &= \partial_\mu \phi^I - \frac{i\lambda}{\mathcal{E}} [A_\mu, \phi^I] \end{aligned}$$

Then we get

$$\mathcal{L}_{kin} = -\lambda^2 \left(F_{\mu a} F^{\mu a} + G_{IJ} D_\mu \phi^I D^\mu \phi^J \right) - 2 T^\alpha T_\alpha A_\mu^- A^{-\mu}$$

7.2 The Chern-Simons term

The Chern-Simons term becomes

$$\mathcal{L}_{CS} = \epsilon^{\mu\nu\lambda} \left\langle A_\mu^- F_{\nu\lambda}^+ - \frac{2i}{3} A_\mu^- A_\nu^- A_\lambda^- \right\rangle$$

where

$$F_{\mu\nu}^+ = \partial_\mu A_\nu^+ - \partial_\nu A_\mu^+ - i[A_\mu^+, A_\nu^+]$$

Alternatively

$$\mathcal{L}_{CS} = -\frac{\lambda}{\mathcal{E}} \epsilon^{\mu\nu\lambda} \left\langle A_\mu^- F_{\nu\lambda} \right\rangle$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{i\lambda}{\mathcal{E}}[A_\mu, A_\nu]$$

7.3 The sextic potential

7.3.1 Quadratic order

From a technical point of view, the ABJM sextic potential is highly complicated to expand. It here advantegous to make use of the BLG formulation instead. It is useful to define a sign

$$s(\alpha) = \pm 1$$

according to whether $\underline{\alpha} = {}^A$ or $\underline{\alpha} = {}_A$. That is, $s({}^A) = 1$ and $s({}_A) = -1$. Then we have

$$\partial_\psi X^\alpha = i s(\alpha) X^\alpha$$

and we have the following useful result,

$$s(\alpha) \partial_m T^\alpha \partial_n T_\alpha = 0$$

Instead of expanding the ABJM sextic potential, we may now instead expand the equivalent BLG sextic potential. At the moment we will be ignorant about the overall factor, which we will determined later. At quadratic order in Y_α we then consider the following terms,

$$\begin{aligned}
\hbar^{-2} V &= \frac{1}{12}\{X^\alpha, X^\beta, X^\gamma\}\{X_\alpha, X_\beta, X_\gamma\} \\
&= \frac{1}{4}\{T^\alpha, T^\beta, Y^\gamma\}\{T_\alpha, T_\beta, Y_\gamma\} + \frac{1}{2}\{T^\alpha, T^\beta, Y^\gamma\}\{T_\alpha, Y_\beta, T_\gamma\} + \frac{1}{2}\{T^\alpha, T^\beta, T^\gamma\}\{T_\alpha, Y_\beta, Y_\gamma\}
\end{aligned}$$

We write this out explicitly as

$$\begin{aligned}
\hbar^{-2} V &= 2 g^{\gamma\gamma'} \partial_\gamma Y^{\underline{\gamma}} \partial_{\gamma'} Y_{\underline{\gamma}} \\
&\quad + \left(g^{\beta\beta'} g^{\gamma\gamma'} - g^{\beta\gamma'} g^{\gamma\beta'}\right) \left(\partial_{\gamma'} T_{\underline{\gamma}} \partial_\gamma Y^{\underline{\gamma}} + \partial_\gamma T_{\underline{\gamma}} \partial_{\gamma'} Y^{\underline{\gamma}}\right) \partial_\beta T^{\underline{\beta}} \partial_{\beta'} Y_{\underline{\beta}}
\end{aligned}$$

where we now start to underline the $SO(8)$ indices $\underline{\alpha}$ in order to distinguish them from the indices α on M_3. We now need the derivatives

$$\begin{aligned}
\partial_\psi Y^{\underline{\alpha}} &= is(\underline{\alpha})Y^{\underline{\alpha}} \\
\partial_m Y^{\underline{\alpha}} &= \partial_m Y^n \partial_n T^{\underline{\alpha}} + \partial_m Y^I \partial_I T^{\underline{\alpha}} + Y^y \partial_m \partial_y T^{\underline{\alpha}}
\end{aligned}$$

We will neglect the last term which is

$$Y^y \partial_m \partial_y T^{\underline{\alpha}} = \mathcal{O}\left(\frac{R}{v}\right)$$

We split the indices as $\underline{\alpha} = (a, \psi)$ and accordingly we split $\hbar^{-2}V = V_{(I)} + V_{(II)} + V_{(III)}$ where

$$\begin{aligned}
V_{(I)} &= 2G^{ab}\partial_a Y^{\underline{\gamma}} \partial_b Y_{\underline{\gamma}} \\
&\quad + \left(G^{ab}G^{cd} - G^{ad}G^{cb}\right)\left(\partial_d T_{\underline{\gamma}} \partial_c Y^{\underline{\gamma}} + \partial_c T_{\underline{\gamma}} \partial_d Y^{\underline{\gamma}}\right) \partial_a T^{\underline{\beta}} \partial_b Y_{\underline{\beta}} \\
V_{(II)} &= 2g^{\psi\psi} Y^{\underline{\gamma}} Y_{\underline{\gamma}} \\
V_{(III)} &= 4G^{ab} g^{\psi\psi} T^{\underline{\alpha}} Y_{\underline{\alpha}} \partial_a T^{\underline{\beta}} \partial_b Y_{\underline{\beta}}
\end{aligned}$$

We have also the mixed terms, which can be gathered into

$$G^{ab} g^{\psi\psi} \partial_a (T^{\underline{\alpha}} Y_{\underline{\alpha}}) \partial_b (T^{\underline{\beta}} Y_{\underline{\beta}}) s(\underline{\alpha}) s(\underline{\beta})$$

and this vanishes by

$$s(\underline{\alpha}) T^{\underline{\alpha}} Y_{\underline{\alpha}} = 0$$

it being understood that this is to be summed over $\underline{\alpha}$. For the remaining pieces we find

$$V_{(III)} = \frac{16\lambda^2}{v^2} G \epsilon^{ab} \epsilon^{cd} v_a A_b \partial_c A_d$$

$$V_{(II)} = \frac{4\lambda^2}{v^2}\left(G^{ab} A_a A_b + G_{IJ}\phi^I \phi^J\right)$$

$$V_{(I)} = 2\lambda^2 f_{ab} f^{ab} + 4\lambda^2 \partial_a \phi^I \partial^a \phi_I + \mathcal{O}\left(\frac{R}{v}\right)$$

We now see that in order to get the kinetic term on the form

$$-\lambda^2 G_{IJ}\left(\eta^{\mu\nu} D_\mu \phi^I D_\nu \phi^J + G^{ab} D_a \phi^I D_b \phi^J\right)$$

we must rescale the scalar fields in BLG theory. A rescaling

$$X^{\underline{\alpha}} \to \mu X^{\underline{\alpha}}$$

yields

$$\mu^2\left(-\frac{1}{2}\left\langle D_\mu X^{\underline{\alpha}}, D^\mu X^{\underline{\alpha}}\right\rangle - \frac{\mu^4}{12}\left\langle [X^{\underline{\alpha}}, X^{\underline{\beta}}; X^{\underline{\gamma}}], [X^{\underline{\alpha}}, X^{\underline{\beta}}; X^{\underline{\gamma}}]\right\rangle\right)$$

Here we shall take

$$\mu^2 = \frac{1}{2\hbar}$$

Then the BLG Lagrangian reads

$$\begin{aligned}&\frac{1}{2\hbar}\left(-\frac{1}{2}\langle D_\mu X^{\underline{\alpha}}, D^\mu X^{\underline{\alpha}}\rangle - \frac{1}{48\hbar^2}\left\langle [X^{\underline{\alpha}}, X^{\underline{\beta}}; X^{\underline{\gamma}}], [X^{\underline{\alpha}}, X^{\underline{\beta}}; X^{\underline{\gamma}}]\right\rangle\right)\\&+\frac{i\lambda}{\mathcal{E}}\epsilon^{\mu\nu\lambda}\left\langle A^-_\mu F_{\nu\lambda}\right\rangle\end{aligned}$$

and we wish to integrate out A^-_μ. We collect terms that contain A^-_μ,

$$\begin{aligned}&\frac{1}{2\hbar}2T^{\underline{\alpha}}T_{\underline{\alpha}}A^-_\mu A^{-\mu} + \frac{i\lambda}{\mathcal{E}}\epsilon^{\mu\nu\lambda}A^-_\mu F_{\nu\lambda}\\ \rightarrow\quad &\left(\frac{\lambda}{\mathcal{E}}\right)^2 \frac{\hbar}{2T^{\underline{\alpha}}T_{\underline{\alpha}}}F_{\mu\nu}F^{\mu\nu}\left(1+\mathcal{O}\left(\frac{R}{v}\right)\right)\end{aligned}$$

We now note that

$$T^{\underline{\alpha}}T_{\underline{\alpha}} = 2T^A T_A = 2v^2\left(1+\mathcal{O}\left(\frac{R}{v}\right)\right)$$

and by re-instating the overall factor and the explicit realization of the inner product as $\frac{KN}{2\pi}\int\frac{d^2\sigma}{4\pi^2}$, we get

$$\int d^2\sigma\sqrt{G}\frac{1}{4\pi^2R^2}\frac{KN}{2\pi}\left(\frac{\lambda}{\mathcal{E}}\right)^2\frac{2\pi R^2 v}{2N2R^2v^2}F_{\mu\nu}F^{\mu\nu}\left(1+\mathcal{O}\left(\frac{R}{v}\right)\right)$$

where we have rewritten the measure in a covariant form and used the fact that $\sqrt{G} = R^2$. Simplifying these factors, we end up with

$$\int d^2\sigma\sqrt{G}\left(\frac{\lambda}{\mathcal{E}}\right)^2\frac{K}{4\pi^2 v}\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\left(1+\mathcal{O}\left(\frac{R}{v}\right)\right)$$

and we can already here read off the Yang-Mills coupling constant as

$$g^2_{YM} = 4\pi^2\frac{v}{K}\left(1+\mathcal{O}\left(\frac{R}{v}\right)\right)$$

Once we now have obtained the correct normalization of the sextic potential, we can now start computing it to various orders in the fluctuation fields.

7.3.2 Zeroth order

At zeroth order we have

$$\begin{aligned}\left\langle [T^{\underline{\alpha}}, T^{\underline{\beta}}; T^{\underline{\gamma}}], [T^{\underline{\alpha}}, T^{\underline{\beta}}; T^{\underline{\gamma}}]\right\rangle &= \hbar^2 g\epsilon^{\alpha\beta\gamma}\epsilon^{\alpha'\beta'\gamma'}8g_{\alpha\alpha'}g_{\beta\beta'}g_{\gamma\gamma'}\langle 1\rangle\\&= 48\hbar^2\langle 1\rangle\end{aligned}$$

Here

$$\langle 1\rangle = \int\frac{d^2\sigma}{(2\pi)^2}1 = 1$$

Now

$$\begin{aligned}\mathcal{L}_{pot} &= -\frac{KN}{2\pi 96\hbar^4}\left\langle[T^{\underline{\alpha}},T^{\underline{\beta}};T^{\underline{\gamma}}],[T^{\underline{\alpha}},T^{\underline{\beta}};T^{\underline{\gamma}}]\right\rangle \\ &= -\frac{KN}{4\pi\hbar^2} \\ &= -\frac{KN^3}{16\pi^3R^4v^2}\end{aligned}$$

7.3.3 Linear order

At linear order we have 6 identitcal terms,

$$\begin{aligned}& 6\left\langle[T^{\underline{\alpha}},T^{\underline{\beta}};T^{\underline{\gamma}}],[T^{\underline{\alpha}},T^{\underline{\beta}};Y^{\underline{\gamma}}]\right\rangle \\ =\ & 6\hbar^2 g\epsilon^{\alpha\beta\gamma}\epsilon^{\alpha'\beta'\gamma'}\partial_\alpha T^{\underline{\alpha}}\partial_\beta T^{\underline{\beta}}\partial_\gamma T^{\underline{\gamma}}\partial_{\alpha'}T_{\underline{\alpha}}\partial_\beta T_{\underline{\beta}}\partial_\gamma Y_{\underline{\gamma}} \\ =\ & 48\hbar^2\lambda\sqrt{G}\epsilon^{ab}f_{ab}\end{aligned}$$

Including the correct overall normalization and the combinatorical factor of 6, we have

$$\mathcal{L}_{pot} = -\frac{\lambda}{2\hbar^2}\sqrt{G}\epsilon^{ab}f_{ab}$$

Now we must also find the non-Abelian completion of this term in the higher order terms.

7.3.4 Quadratic order

The terms that we previously overlooked at quadratic order are given by (there are 6 terms of this type)

$$\begin{aligned}& \left\langle[T^{\underline{\alpha}},T^{\underline{\beta}},T^{\underline{\gamma}}],[T^{\underline{\alpha}},Y^{\underline{\beta}};Y^{\underline{\gamma}}]\right\rangle \\ =\ & -8i\mathcal{E}v^2\lambda^2\sqrt{G}\epsilon^{ab}[A_a,A_b]\end{aligned}$$

This combines with the linear order term into

$$8\hbar^2\lambda\sqrt{G}\epsilon^{ab}F_{ab}$$

where

$$F_{ab} = \partial_a A_b - \partial_b A_a - \frac{i\lambda}{\mathcal{E}}[A_a,A_b].$$

We have from earlier computation

$$\langle[XXX],[XXX]\rangle|_{quadratic} = 24\hbar^2\lambda^2 f_{ab}f^{ab}$$

which combines with the linear and zeroth order terms into

$$\begin{aligned}& 48\hbar^2\left(1+\lambda\sqrt{G}\epsilon^{ab}f_{ab}+\frac{1}{2}\lambda^2 f^{ab}f_{ab}\right) \\ =\ & 24\hbar^2\lambda^2\left(f_{ab}+\lambda^{-1}\sqrt{G}\epsilon_{ab}\right)\left(f^{ab}+\lambda^{-1}\sqrt{G}\epsilon^{ab}\right)\end{aligned}$$

Normalizing and rewriting the trace form as

$$\langle ...\rangle = \frac{1}{4\pi^2R^2}\int d^2\sigma\sqrt{G}$$

we have

$$-\frac{KN}{2.96\pi\hbar^3}\frac{1}{4\pi^2R^2}\int d^2\sigma\sqrt{G}[XXX][XXX] \quad = \quad -\left(\frac{\lambda}{\mathcal{E}}\right)^2\frac{K}{16\pi^2 v}\int d^2\sigma\sqrt{G}\widetilde{f}_{ab}\widetilde{f}^{ab}$$

where

$$\widetilde{f}_{ab} \quad = \quad f_{ab}+\lambda^{-1}\sqrt{G}\epsilon_{ab}$$

7.3.5 Cubic interactions

First we note that there are no contributions on the form

$$\left\langle [T^{\underline{\alpha}},T^{\underline{\beta}},T_{\underline{\gamma}}]_{\mathcal{A}},[Y^{\underline{\alpha}},Y^{\underline{\beta}};Y^{\underline{\gamma}}]_{\mathcal{B}}\right\rangle$$

Using the three-algebra satisfied by the $T^{\underline{\alpha}}$, we can write this as a sum of terms each of the form

$$\left\langle T^{\beta},[Y^{\alpha},Y^{\beta};Y^{\alpha}]\right\rangle$$

Now this vanishes due to $T^{\underline{\alpha}}Y_{\underline{\alpha}}=0$.

We next note that

$$[X^{\underline{\alpha}},X^{\underline{\beta}};X^{\underline{\gamma}}] \quad = \quad X_{\underline{\gamma}}[X^{\underline{\alpha}},X^{\underline{\beta}}]_{s(\gamma)}+[X^{\underline{\alpha}},X^{\underline{\beta}}]_{s(\beta)}X^{\underline{\beta}}-[X^{\underline{\beta}},X_{\underline{\gamma}}]_{s(\underline{\alpha})}X^{\underline{\alpha}}$$

where the subscripts indicate which sign to use for the non-commutativity parameter, that is $s(\underline{\alpha})\mathcal{E}$, in the star-commutator. We keep the following terms

$$\begin{aligned}
[T^{\underline{\alpha}},T^{\underline{\beta}};Y^{\underline{\gamma}}] &= [T^{\underline{\alpha}},Y_{\underline{\gamma}}]_{s(\underline{\beta})}T^{\underline{\beta}}-[T^{\underline{\beta}},Y_{\underline{\gamma}}]_{s(\underline{\alpha})}T^{\underline{\alpha}}\\
[T^{\underline{\alpha}},Y^{\underline{\beta}};Y^{\underline{\gamma}}] &= -[Y^{\underline{\beta}},Y_{\underline{\gamma}}]_{s(\alpha)}T^{\underline{\alpha}}
\end{aligned}$$

and we get

$$\left\langle [T^{\underline{\alpha}},T^{\underline{\beta}};Y^{\underline{\gamma}}],[T^{\underline{\alpha}},Y^{\underline{\beta}};Y^{\underline{\gamma}}]\right\rangle \quad = \quad 8iv^2\mathcal{E}\lambda^3G^{ab}\left(G^{cd}\left\langle\partial_aA_c,[A_b,A_d]\right\rangle+G_{IJ}\left\langle\partial_a\phi^I,[A_b,\phi^J]\right\rangle\right)$$

7.3.6 Quartic interaction

We only get quartic contributions from terms which are on the form

$$\begin{aligned}
&\left\langle [T^{\underline{\alpha}},Y^{\underline{\beta}};Y^{\underline{\gamma}}],[T^{\underline{\alpha}},Y^{\underline{\beta}};Y^{\underline{\gamma}}]\right\rangle\\
=\;& 8v^2\lambda^4\left\langle[\phi^I,\phi^J][\phi_I,\phi_J]+2[A_a,\phi^I][A^a,\phi_I]+[A_a,A_b][A^a,A^b]\right\rangle
\end{aligned}$$

7.4 Summarizing

We shall multiply the cubic term by a combinatorical factor of 12 and the quartic term by a combinatorical factor of 3. By collecting the terms, we have now found

$$\begin{aligned}
\langle[XXX],[XXX]\rangle_{quadratic} &= 24\hbar^2\lambda^2\left(\widetilde{f}_{ab}\widetilde{f}^{ab}+\partial_a\phi^I\partial^a\phi_I\right)\\
\langle[XXX],[XXX]\rangle_{cubic} &= 12.8iv^2\mathcal{E}\lambda^3\left(\frac{1}{2}\widetilde{f}_{ab}[A^a,A^b]+\partial_a\phi^I[A^a,\phi_I]\right)\\
\langle[XXX],[XXX]\rangle_{quartic} &= 3.8v^2\lambda^4\left([\phi^I,\phi^J][\phi_I,\phi_J]+2[A_a,\phi^I][A^a,\phi_I]+[A_a,A_b][A^a,A^b]\right)
\end{aligned}$$

Summing these terms we see that we obtain gauge covariant expressions. Reinstating the correct overall normalization, the total Lagrangian we have obtained up to zeroth order in $\frac{R}{v}$ is given by

$$\begin{aligned}
&-\frac{1}{4g_{YM}^2}\Bigg(\frac{1}{4}\left(\widetilde{F}_{ab}\widetilde{F}^{ab}+2F_{\mu a}F^{\mu a}+F_{\mu\nu}F^{\mu\nu}\right)\\
&+\frac{1}{2}\left(D_a\phi^I D^a\phi_I+D_\mu\phi^I D^\mu\phi^I\right)\\
&+\frac{1}{4}[\phi_I,\phi_J][\phi^I,\phi^J]\\
&+\frac{1}{2v^2}\left(G^{ab}A_aA_b+G_{IJ}\phi^I\phi^J+4G\epsilon^{ab}\epsilon^{cd}v_aA_b\partial_cA_d\right)\Bigg)
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{F}_{ab} &= F_{ab}+\mathcal{E}^{-1}\sqrt{G}\epsilon_{ab}\\
F_{ab} &= \partial_aA_b-\partial_bA_a-i[A_a,A_b]\\
F_{\mu\nu} &= \partial_\mu A_\nu-\partial_\nu A_\mu-i[A_\mu,A_\nu]\\
F_{\mu a} &= \partial_\mu A_a-\partial_a A_\mu-i[A_\mu,A_a]\\
D_a\phi^I &= \partial_a\phi^I-i[A_a,\phi^I]\\
D_\mu\phi^I &= \partial_\mu\phi^I-i[A_\mu,\phi^I]
\end{aligned}$$

and

$$g_{YM}^2 = 4\pi^2\frac{v}{K}$$

Here we have thus rescaled the field so as to absorb the factor of $\frac{\mathcal{E}}{\lambda}$.

Since we have generated gauge variant mass term for the gauge potential at order $\frac{1}{v}$ we see that our approximation where we cut out from a curved three-manifold, a flat two-torus, breaks down at this order. We should not trust this Lagrangian to this first order in $\frac{1}{v}$.

The inner product on the D4 brane Lagrangian has been suppressed. The inner product in ABJM theory is given by

$$\frac{1}{N}\mathrm{tr} = \frac{1}{(2\pi)^2}\int d^2\sigma$$

and we decompose

$$N = N_{\mathcal{A}}N_{\mathcal{B}}$$

following [4]. Then N counts the number of M2 branes, $N_{\mathcal{B}}$ counts the number of $D4$ branes. After tracing over $\mathcal{A}$ indices, the residual the inner product on the D4 brane is over $\mathcal{B}$ indices,

$$\mathrm{tr}_{\mathcal{B}}$$

and is unit normalized since we start with the ABJM inner product which we decompose as

$$\frac{1}{N_{\mathcal{A}}N_{\mathcal{B}}}\mathrm{tr} = \mathrm{tr}_{\mathcal{B}}\frac{1}{N_{\mathcal{A}}N_{\mathcal{B}}}\mathrm{tr}_{\mathcal{A}}$$

Hence by mapping the star-product to matrix product in the D4 Lagrangian above, the inner product to be used is precisely $\mathrm{tr}_{\mathcal{B}}$.

8 Single M5

We saw that the D4 brane Lagrangian we obtained is not gauge covariant. To take proper care of the three-manifold which is rather invisible from D4 brane point of view, we will now instead consider the single M5 brane. Let us use real embedding coordinates X^I as originally was used in BLG theory, for clarity. We then expand in fluctuations as

$$Y^I = Y^\alpha \partial_\alpha T^I$$

and ignore the scalar fields for the time being. Since M_3 is curved it is essential that we use covariant derivatives when computing the derivative

$$\partial_\alpha Y^I = D_\alpha Y^\beta \partial_\beta T^I + Y^\beta D_\alpha \partial_\beta T^I$$

Eventually we shall dualize

$$Y^\alpha = \frac{1}{2}\sqrt{g}\epsilon^{\alpha\beta\gamma} B_{\beta\gamma}$$

Our main goal now, is to in particular show that no gauge variant mass term $g_{\alpha\beta} Y^\alpha Y^\beta$ for the gauge potential arises when proper care is taken of the M5 brane geometry.

We expand the sextic potential to quadratic order. Let us here define the metric and the second fundamental form as

$$\begin{aligned} g_{\alpha\beta} &= \partial_\alpha T^I \partial_\beta T^I \\ \Omega^I_{\alpha\beta} &= D_\alpha \partial_\beta T^I \end{aligned}$$

By using the metric compatibility condition and the torsion free condition

$$\begin{aligned} D_\gamma g_{\alpha\beta} &= 0 \\ \Omega^I_{\alpha\beta} &= \Omega^I_{\beta\alpha} \end{aligned}$$

we obtain at quadratic order

$$-\frac{1}{12}\{X,X,X\}\{X,X,X\}|_{quadratic} = -\frac{1}{24}\left(3(D_\alpha Y^\alpha)^2 + M_{\alpha\beta} Y^\alpha Y^\beta - Y^\alpha [D_\alpha, D_\beta] Y^\beta\right)$$

where

$$M_{\alpha\beta} = g^{\gamma\delta} \Omega^I_{\alpha\gamma} \Omega^I_{\beta\delta}$$

The first term is what we want. Upon dualizing it becomes

$$-\frac{3}{24}(D_\alpha Y^\alpha)^2 = -\frac{1}{12} H_{\alpha\beta\gamma} H^{\alpha\beta\gamma}$$

where

$$H_{\alpha\beta\gamma} = D_\alpha B_{\beta\gamma} + D_\gamma B_{\alpha\beta} + D_\beta B_{\gamma\alpha}$$

The other two terms are unwanted and could give rise to gauge variant mass terms for the gauge potential. We will now demonstate the cancelation

$$Y^\alpha \left(M_{\alpha\beta} - [D_\alpha, D_\beta]\right) Y^\beta = 0 \tag{6}$$

We define the curvature tensor according to

$$[D_\alpha, D_\beta]V^\gamma = R^\gamma{}_{\tau\alpha\beta}V^\tau$$

We may use the Gauss-Codazzi relation[3]

$$R_{\delta\gamma\alpha\beta} = \Omega^I_{\alpha\delta}\Omega^I_{\beta\gamma} - \Omega^I_{\alpha\gamma}\Omega^I_{\beta\delta}$$

from which follows that

$$R_{\alpha\beta} = M_{\alpha\beta} - \Omega^I_{\alpha\beta}\Omega^I$$

where we define

$$\begin{aligned} \Omega^I &= g^{\alpha\beta}\Omega^I_{\alpha\beta} \\ R_{\alpha\beta} &= R^\gamma{}_{\alpha\beta\gamma} \end{aligned}$$

We will now compute Ω^I explicitly for our specific three-manifold M_3. We switch to the complex basis where only T^a are non-vanishing, and we compute

$$\Omega^a_{\alpha\beta} = \frac{1}{\sqrt{g}}\partial_\alpha\left(\sqrt{g}g^{\alpha\beta}\partial_\beta T^a\right)$$

for the embedding

$$T^a = x^a e^{i\psi}$$

and define $r = \sqrt{x^a x^a}$. Then

$$\begin{aligned} \Omega^b &= \frac{1}{r}\partial_a r\partial_a T^b + \partial_a\partial_a T^b + \frac{1}{r^2}\partial_\psi^2 T^b \\ &= \left(\frac{x^b}{r^2} - \frac{x^b}{r^2}\right)e^{i\psi} \\ &= 0 \end{aligned}$$

for our specific three-manifold. By finally noting that

$$[D_\alpha, D_\beta]Y^\beta = R_{\alpha\beta}Y^\beta$$

we see that the cancelation (6) does occur. No gauge variant mass term for the gauge potential is generated.

8.1 Direct derivation of the M5 brane coupling constant

We have obtain the M5 brane coupling constant in a rather indirect way by obtaining the SYM coupling constant $g_{YM}^2 = 4\pi^2 v/K$, which is what we get by dimensional reduction of M5 brane on a circle of radius v/K. But it would also be nice if we could determine the M5 brane coupling constant directly by constructing the abelian M5 brane including the right normalization. We will address this problem here.

After rescaling the scalar fields, our starting point is the Lagrangian

$$\frac{KN}{2\pi\hbar}\left(-\frac{1}{2}\left\langle D_\mu X^I, D^\mu X^I\right\rangle - \frac{1}{12}\left\langle\{X^I, X^J, X^K\}, \{X^I, X^J, X^K\}\right\rangle\right)$$

[3]General expressions are found in [8].

where we focus only on the scalar fields. That will be sufficient for our purpose of determining the overall M5 brane coupling constant. In order to have a finite $\hbar$ we need to discretize the space. Again we do this by taking out a local two-torus at some large v with radii R. Then we get as before

$$\hbar = \frac{2\pi R^2 v}{N}$$

Let us make the following ansatz for the fluctuation fields

$$Y^\alpha = \lambda \frac{1}{2}\sqrt{g}\epsilon^{\alpha\beta\gamma}B_{\beta\gamma}$$

and we express the inner product as

$$\begin{aligned}
\langle\rangle &= \frac{1}{N}\mathrm{tr} \\
&= \int \frac{d^2\sigma}{(2\pi)^2} \\
&= K\int \frac{d^2\sigma d\psi}{(2\pi)^3} \\
&= \frac{K}{(2\pi)^3 R^2 v}\int d^3\sigma\sqrt{g}
\end{aligned}$$

where $\sqrt{g} = R^2 v$ if the metric is $ds^2 = R^2 d\sigma^a d\sigma^a + v^2 d\psi^2$. We now see that by taking

$$\lambda = \frac{4\pi^2 R^2 v}{KN} \tag{7}$$

the sextic potential gives the contribution

$$\frac{1}{4\pi}\left(-\frac{1}{6}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma}\right)$$

to the full M5 brane Lagrangian. Other components of the three-form field strength, H_{MNP} where $M = (\mu, \alpha)$, come from the Chern-Simons term and the kinetic term. But not even by taking all these contributions into account we get a fully Lorentz covariant expression. This is of course to be expected since H_{MNP} is supposed to be selfdual and no Lorentz covariant action exists. But this problem is well-known and in the present case it has been analysed in [9].

We obtain the correct normalization of the M5 brane Lagrangian for a two-form connection H which is subject to the Dirac charge quantization

$$\int_{3-cycle}\frac{1}{6}dx^M\wedge dx^N\wedge dx^P H_{MNP} \in 2\pi\mathbb{Z}$$

by choosing λ as in (7). Now it remains to explain this specific choice of λ. A natural three-cycle to consider in the present situation, is the two-torus over which we have a circle fiber. Then we have

$$\int d^3\sigma\sqrt{g}\partial_\alpha Y^\alpha = \lambda\int d^3\sigma\frac{1}{6}g\epsilon^{\alpha\beta\gamma}H_{\alpha\beta\gamma}$$

Now we have defined

$$Y^I = Y^\alpha \partial_\alpha T^I$$

it is natural to identify Y^α with a reparametrization of the torus,

$$\sigma^\alpha \to \sigma^\alpha + Y^\alpha$$

and then we must require

$$Y^\alpha(2\pi) = Y^\alpha(0) + 2\pi w^\alpha$$

where w^α are integer winding numbers. Then we get

$$\int d^3\sigma \sqrt{g} \partial_\alpha Y^\alpha = (2\pi)^3 R^2 v \left(\frac{w^1}{K} + \frac{w^2}{K} + w^3\right)$$

and from the Dirac quantization condition, we conclude that

$$\lambda = \frac{4\pi^2 R^2 v}{K}$$

This misses out one factor of N in the denominator. To get it, we would need the magnetic charge to be quantized as

$$\int H = 2\pi N \mathbb{Z}$$

Even though a similar magnetic charge has been observed for a compact D2 brane bound to N D0 branes [10], it appears to be different anyway. For N D0 branes bound to a D2, the magnetic charge is given by N and not $N\mathbb{Z}$.

9 Discussion and outlook

We have taken the infinite-N solution of ABJM theory and discretized it by taking a small two-torus piece out of it. It might also be possible to consider the discretized solution which we may derive directly from ABJM theory, but for which only a series expansion in $1/v$ is presently known [1]. Another issue we have largely omitted to discuss is the stability and possible supersymmetric extensions of this solution. One may also ask how to obtain SYM theories with other gauge groups than $U(N)$ gauge groups, from ABJM theories. This question seems to require a more general understanding of star-products and how they can be mapped into matrix multiplications.

Acknowledgements

I would like to thank Futoshi Yagi for discussions. This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST) through the Center for Quantum Spacetime(CQUeST) of Sogang University with grant number 2005-0049409.

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